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Study of Torsional Vibrations in Composite Transversely Isotropic Poroelastic Cylinders

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Abstract

Employing theory of poroelasticity, torsional vibrations in composite transversely isotropic poroelastic solid cylinder are investigated. Equations of motion are presented which are derived from the pertinent equations of equilibrium. Poroelastic composite cylinder here consists of two concentric cylindrical layers made of different poroelastic materials. Phase velocity is computed as a function of wave number and ratio of radii at various anisotropic ratios. The limiting cases of a poroelastic thick-walled hollow cylinder, thin poroelastic shell and plate are discussed. The results are presented graphically for two poroelastic composite cylinders and then discussed.

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Keywords: Poroelasticity; Torsional vibrations; Composite poroelastic cylinder; Frequency equation; Phase velocity; Wavenumber; Ratio of radii

1. Introduction

The study of torsional vibrations in poroelastic solids have many theoretical and practical applications in several fields such as Seismology, Acoustics, Geophysics, Soil-mechanics, Bio-mechanics, Bone mechanics, Civil engineering, and Mechanical engineering. The developments in the field of composite materials are increased from the last quarter of the century due to the breadth and universality of applications. Composite materials have progressed from almost an engineering curiosity to a widely used material in aerospace applications, medical applications as well as many other applications in everyday life. From the real time experiences, one may find that buildings, bridges and some manmade structures consist two or more material that could be combined to take

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advantage of the good characteristics of each of the materials. The theory of linear isotropic poroelasticity was developed by Biot [1]. In the frame work of Biot's theory, the effect of boundaries on torsional vibrations in a poroelastic composite cylinder is reported in several papers [2, 3]. Axially symmetric vibrations of composite poroelastic cylinder are investigated by Malla Reddy and Tajuddin [4]. Studies of torsional vibrations in isotropic poroelastic cylinder are reported in several papers [5, 6, 7]. The analytical solutions for pore pressure and stress fields for inclined bore hole and the cylinder are investigated in the case of transversely isotropic poroelastic solids by Abousleiman and Cui [8]. Axially and non-axially symmetric vibrations of thick walled hollow poroelastic cylinders are investigated by Malla Reddy and Tajuddin [9]. Poromechanic analysis of a fully saturated transversely isotropic hollow cylinder is made by Kanj et. al [10]. The poroelastic behavior of rocks is often more complicated than the common assumptions of linearity and isotropy. The Berea sandstone and shale rock exhibit transversely isotropic behavior at low effective stresses [11, 12]. On Bone mechanics front, Cowin [13] described fluid flow in bone tissues employing the poroelasticity theory. Radial vibrations in thick walled transversely isotropic poroelastic cylindrical bone in presence of dissipation are studied by Malla Reddy and Sandhya Rani [14]. Torsional vibrations in thick-walled cylindrical bone in the framework of transversely isotropic poroelasticity are investigated by Sandhya Rani and Malla Reddy [15]. In the present paper, torsional vibrations in composite transversely isotropic poroelastic solid cylinder of infinite extent are investigated. This composite poroelastic cylinder consists of an inner solid circular cylinder of one material enveloped by another thick-walled hollow cylinder made of another poroelastic material. Phase velocity is computed as a function of wavenumber and ratio of radii at various anisotropic ratios.

The rest of the paper is organized as follows. In section 2, governing equations and solution of the problem are discussed. Boundary conditions and frequency equation are presented in section 3. Particular cases are given in section 4. Numerical results are discussed in section 5. Finally, conclusions are given in section 6.

2. Governing equations and solution of the problem

Consider cylindrical coordinate system (r,θ,z) with z - axis along the axis of poroelastic cylinder. Let the cylinder be homogeneous and transversely isotropic. Assume that z - axis is along the axis of rotational symmetry. Hence, in this case eight constants are involved. The constitutive stress-strain relations for transversely isotropic poroelastic solid [8] are given under.

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{r\theta}\\ \tau_{\theta z}\\ \tau_{rz}\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12} & M_{13} & 0 & 0 & 0\\ M_{12} & M_{11} & M_{13} & 0 & 0 & 0\\ M_{13} & M_{13} & M_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & M_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & M_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & M_{55}\end{bmatrix}\begin{bmatrix}e_{rr}\\ e_{\theta\theta}\\ e_{zz}\\ \gamma_{r\theta}\\ \gamma_{\theta z}\\ \gamma_{rz}\end{bmatrix}-\begin{bmatrix}\alpha\\ \alpha\\ \alpha'\\ 0\\ 0\\ 0\end{bmatrix}p,$$
$$p = M[\varepsilon-\alpha(e_{rr}+e_{\theta\theta})-\alpha' e_{zz}]. \tag{1}$$

In the Eq. (1), e_{jj} and γ_{jk} are normal and shear strain components; σ_{jj} and τ_{jk} are normal and shear stress components in the cylindrical coordinate system, respectively [8]; p is the pore pressure, α and α' are Biot's effective stress coefficients in the isotropic plane (r - θ plane) and in the z - direction, respectively; M is Biot's modulus, ε is the variation of fluid content per unit reference volume, and M_{jk} are components of the drained elastic modulus which depend on E, E', ν ν', G and G'. E and ν are drained Young's modulus and Poisson ratio in the isotropic plane, E' and ν' are similar quantities as that of E and ν pertaining to the direction of the axis of symmetry, G and G' are the shear modulus related to the direction of the isotropic plane and axis of symmetry, respectively. For given anisotropic ratios of $N_E=\frac{E'}{E}$ and $N_\nu=\frac{\nu'}{\nu}$, E' and ν' can be determined. Different

N_E and N_ν ratios define different degrees of anisotropy.

Consider a composite concentric isotropic infinite poroelastic solid cylinder with inner and outer radii r_1 and r_2, respectively. The z-axis coincides with the axis of the cylinder. The substrate is a circular solid cylinder with radius r_1 and the coating is a thick-walled hollow cylinder having thickness $h\ (= r_2 - r_1)$. Non - zero displacement components of solid and fluid for torsional vibrations are $\vec{u}(0,u_2,0)$ and $\vec{U}(0,U_2,0)$, respectively. The equations of motion of transversely isotropic poroelastic solids in presence of dissipation (b) in the case of torsional vibrations are

$$2M_{44}\frac{\partial^2 u_2}{\partial r^2}+2M_{55}\frac{\partial^2 u_2}{\partial z^2}+\frac{M_{44}}{r}\frac{\partial u_2}{\partial r}-\frac{2M_{44}u_2}{r^2}=\frac{\partial^2}{\partial t^2}(\rho_{11}u_2+\rho_{12}U_2)+b\frac{\partial}{\partial t}(u_2-\mathrm{U}_2),$$
$$0=\frac{\partial^2}{\partial t^2}(\rho_{12}u_2+\rho_{22}U_2)-\mathrm{b}\frac{\partial}{\partial \mathrm{t}}(u_2-\mathrm{U}_2). \tag{2}$$

In Eq. (2), ρ_{ij} are mass coefficients. In the case of harmonic waves, the solutions for circumferential displacements are assumed as follows:

$${}_ju_2(r,z,t)={}_jf(r)\,\mathrm{e}^{\mathrm{i}(kz+\omega t)},\quad {}_jU_2(r,z,t)={}_jF(r)\,\mathrm{e}^{\mathrm{i}(kz+\omega t)},\quad j=1,2. \tag{3}$$

In Eq. (3), the quantities with subscripts 1 refer to the substrate while 2 refers to the coating. ω is the frequency of wave, k is the wavenumber, i is complex unity, and t is time. Substituting Eq. (1) and Eq. (3) in Eq. (2), one obtains

$$2\mathrm{M}_{44}\frac{d^2\,{}_jf}{dr^2}+\frac{2\mathrm{M}_{44}}{r}\frac{d\,{}_jf}{dr}-(2k^2M_{55}+\frac{2\mathrm{M}_{44}}{r^2})\,{}_jf=-\omega^2[(\rho_{11}-\frac{ib}{\omega})\,{}_jf+(\rho_{12}+\frac{ib}{\omega})\,{}_jF],$$
$$0=-\omega^2[(\rho_{12}+\frac{ib}{\omega})\,{}_jf+(\rho_{22}-\frac{ib}{\omega})\,{}_jF]. \tag{4}$$

Solutions of Eq. (4) are

$${}_2f(r)=C_1J_1({}_2qr)+C_2Y_1({}_2qr),\quad {}_1f(r)=C_3J_1({}_1qr). \tag{5}$$

In Eq. (5), ${}_jq=k(\frac{-{}_jM_{55}}{{}_jM_{44}}+c^2(\frac{{}_j\rho_{11}\,{}_j\rho_{22}-{}_j\rho_{12}^2}{2\,{}_jM_{44}\,{}_j\rho_{22}}))^{\frac{1}{2}},\quad j=1,\ 2,$

where, c is the torsional waves velocity. The non-zero stresses both for the substrate and the coating are

$${}_2\sigma_{r\theta}=-{}_2N\,{}_2q\left(\mathrm{C}_1J_2\left({}_2qr\right)+C_2Y_2\left({}_2qr\right)\right)\mathrm{e}^{\mathrm{i}k(\mathrm{z+ct})},$$
$${}_1\sigma_{r\theta}=-{}_1N\,{}_1q\left(\mathrm{C}_3J_2\left({}_1qr\right)\right)\mathrm{e}^{\mathrm{i}k(\mathrm{z+ct})}. \tag{6}$$

3. Boundary conditions and frequency equations

The boundary condition for stress free outer surface $r=r_2$ is

$${}_2\sigma_{r\theta}=0\quad \text{at}\quad r=r_2, \tag{7}$$

The conditions for the substrate and the coating at the interface $r=r_1$ are

$${}_2\sigma_{r\theta}={}_1\sigma_{r\theta}\quad \text{at}\quad r=r_1,$$
$${}_2u_2={}_1u_2\quad \text{at}\quad r=r_1. \tag{8}$$

Substitution of Eqs. (3), (5) and (6) in Eqs. (7) and (8) gives three homogeneous equations in three unknowns C_1, C_2, and C_3. A non-trivial solution can be obtained if the determinant of the coefficient matrix vanishes. Accordingly, one obtains the following frequency equation:

$$\left|A_{ij}\right| = 0, \qquad (i, j = 1, 2, 3) \tag{9}$$

where,

$$A_{11} = \frac{-2\,{}_2N}{r_2} J_1({}_2qr_2) + {}_2N\,{}_2q\,J_0({}_2qr_2), \quad A_{12} = \frac{-2\,{}_2N}{r_2} Y_1({}_2qr_2) + {}_2N\,{}_2q\,Y_0({}_2qr_2),$$

$$A_{13} = 0, \quad A_{23} = \frac{-2\,{}_1N}{r_1} J_1({}_1qr_1) - {}_1N_1 q\,J_0({}_1qr_1), A_{31} = J_1({}_2qr_1), \quad A_{32} = Y_1({}_2qr_1), \quad A_{33} = -J_1({}_1qr_1),$$

A_{21}, A_{22} are similar expressions as A_{11}, A_{12} with r_2 replaced by r_1.

4. Particular cases

4.1. Poroelastic thick-walled hollow cylinder

When the material constants of a substrate vanish, the composite poroelastic cylinder will become a thick-walled hollow poroelastic cylinder. Accordingly, we set ${}_1N = 0,\ {}_2N = N, {}_2q = q,\ {}_1q = {}_2q$ at the interface $r = r_1$, then the frequency equation (9) reduces to

$$J_2(qr_2)\,Y_2(qr_1) - J_2(qr_1)\,Y_2(qr_2) = 0, \tag{10}$$

which is the frequency equation of torsional vibrations in poroelastic thick-walled hollow cylinder similar to that of [15].

4.1.1. For thin poroelastic shell

When $\frac{h}{r_1} << 1$ (i.e., $qr_1, qr_2 >> 1$), the asymptotic approximation for Bessel functions [16] can be used. Then the frequency equation (10) reduces to

$$\left(1 - \frac{225}{64q^2 r_1 r_2}\right)\sin qh - \frac{15qh}{8q^2 r_1 r_2}\cos qh \approx 0 \tag{11}$$

Eq. (11) is the frequency equation for a thin cylindrical poroelastic shell. In the limiting case, when $qr_1 \to \infty, qr_2 \to \infty,$ Eq. (11) simplifies to

$$\sin qh = 0, \tag{12}$$

Then Eq. (12) gives

$$c = \left(\frac{M_{44}\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\left(\frac{n^2\pi^2}{h^2k^2} + \frac{2k^2 M_{55}}{M_{44}}\right)\right)^{\frac{1}{2}}, n = 1, 2, 3, \ldots \tag{13}$$

which are phase velocities of poroelastic plate of thickness h. Furthermore, near the origin $hr_1^{-1} = 0$, and substituting

$$qh = n\pi + \varepsilon^*, (\varepsilon^* << 1), \tag{14}$$

into the frequency equation of torsional vibrations of a thin poroelastic cylindrical shell, Eq. (11) gives

$$\varepsilon^* \approx (hr_1^{-1})^2\left(\frac{15n\pi}{(8n\pi)^2 + 210(hr_1^{-1})^2}\right), \quad n = 1, 2, 3, \ldots.. \tag{15}$$

Substituting Eq. (15) into (14) gives the phase velocity values obtained from Eq. (13) in the form

$$c \approx \left(\frac{M_{44}\rho_{22}}{(\rho_{11}\rho_{22} - \rho_{12}^2)h^2k^2}\left(n^2\pi^2\left(1 + (hr_1^{-1})^2\frac{15n\pi}{(8n\pi)^2 + 210(hr_1^{-1})^2}\right) + \frac{2h^2k^2M_{55}}{M_{44}}\right)\right)^{\frac{1}{2}}, n = 1, 2, 3, \ldots. \tag{16}$$

These are the phase velocities of torsional vibrations of a poroelastic plate of thickness h near the origin.

5. Numerical results

Due to presence of dissipative nature of the solids, waves are attenuated. Attenuation presents some difficulty in the definition of phase velocity. If the dissipation coefficient is non-zero, the wavenumber, densities are complex. Consequently velocities of dilatational waves and shear waves are complex valued. Finally, frequency equations will be complex valued and implicit. Therefore, b is made to be zero so that frequency equation will be real valued and the roots will be obtained. Even if b is zero, problem would be poroelastic in nature as the parameters would not vanish. The frequency equation Eq. (9) is investigated numerically for two types of composite poroelastic cylinders, namely, composite cylinder 1 and composite cylinder 2. Composite cylinder 1 consists of the core made of Berea sandstone and the casing with shale rock; while second one is resulted when a cylindrical bone is implanted with Titanium. The material constants M_{44}, M_{55} involves E, E', ν and ν'. The values of Young's modulus (E), Poisson's ratio (ν) for Berea sandstone and shale rock are taken to be 14.4 Gpa, 0.20 and 1.854 Gpa, 0.22 as suggested in the papers [11,12]. Employing these values in the frequency equation, we obtain implicit relation between phase velocity (c), wavenumber (k) and the ratio $(g = r_2 r_1^{-1})$. The phase velocity against wavenumber is computed at various anisotropic ratios in the cases of thin coating $(g = 1.01)$ and thick coating $(g = 4)$. The numerical values are shown in Figs.1-2. Figs. 1-2 depicts the variation of phase velocity against wavenumber when the anisotropic ratio $N_E = 1$ at various N_ν values and $N_\nu = 1$ at various N_E values in the cases of $g = 1.01, 4$. From this figures, it is seen that the phase velocity values increase as g increases. Also, it is observed that phase velocity values depends on the anisotropic ratios.

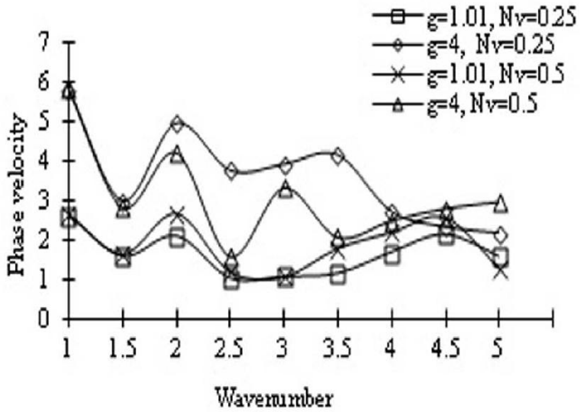

Fig. 1. Variation of phase velocity with wavenumber when $N_E = 1$.

Bone rotatary systems, for example, ball-socket joints (scapula) and anckle joints in human body experience torsional vibrations. Excessive torsional vibrations result in fracture of bones. Participation in the sports that expose joints to torsional loading increase the risk of joint degeneration. Fractured bone is in general implanted with Titanium, then we have a composite cylinder consisting of two different materials one is bone and other is Titanium. The natural selection of Titanium is obvious for its favorable characteristics including immunity to corrosion, bio-compatibility, and the capacity for joining with bone, which is osseointegration. For the said reason, we investigate bone implanted with Titanium. Its density is $4.5 \times 10^3 kg/m^3$. Its Young's modulus and Poisson ratio are 105 Gpa and 0.32, respectively. The values of Young's modulus and Poisson ratio are taken to be 20.684 Gpa and 0.28, respectively as suggested in the paper [17]. Mass coefficients of solid part and fluid are taken to be $1.7634 \times 10^3 kg/m^3$ and $1.4961 \times 10^2 kg/m^3$, respectively, and mass coupling parameter is taken to be zero as in the paper [17]. In the paper [18], it is assumed that the anisotropic ratios of bone vary in the neighbourhood of 1. Fig. 3 depicts the variation of phase velocity against wavenumber when the anisotropic ratio $N_E, N_\nu = 0.98$ and $g = 1.01, 4$. From this figures, it is seen that the phase velocity values increase as g increases.

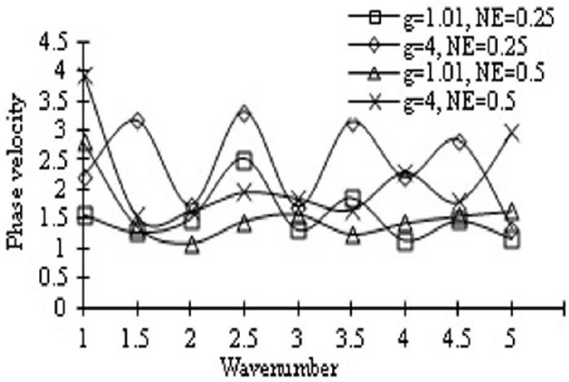

Fig. 2. Variation of phase velocity with wavenumber when $N_\nu = 1$.

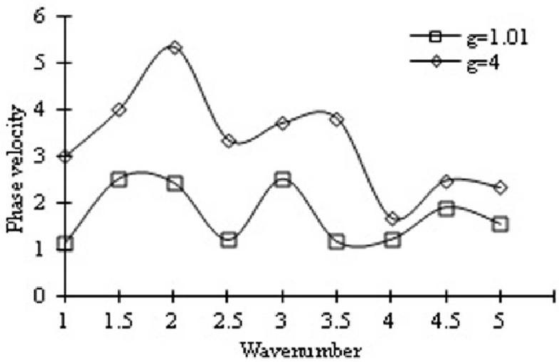

Fig. 3. Variation of phase velocity with wavenumber when $N_E, N_\nu = 0.98$.

6. Conclusion

Torsional vibrations of composite transversely isotropic poroelastic solid cylinder are investigated. Phase velocity is computed as a function of wavenumber and ratio of radii at various anisotropic ratios. The limiting cases are studied by using appropriate approximations. The numerical results were computed at the basis of relevant material data. From the numerical results, one can infer that the phase velocity values are affected with anisotropic ratios. Similar analysis is made for cylindrical bone implanted with Titanium. The numerical results reveal that the phase velocity increase as ratio of radii increases.

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